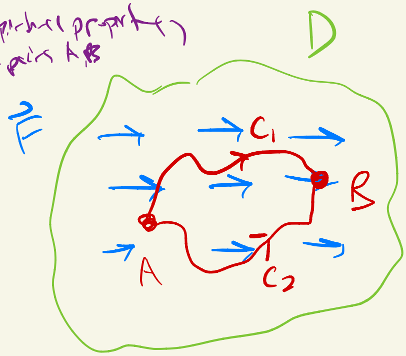
8-25

Last time...

(path-independence property for all pairs A, B)

$\vec{F}$ conservative on D

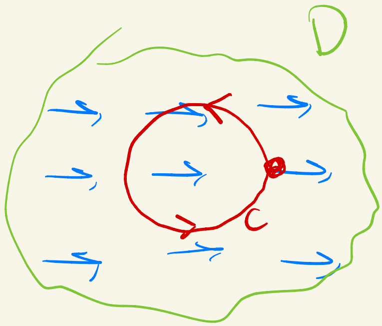


$$\int_{C_1} \vec{F}\cdot d\vec{r}_1 = \int_{C_2} \vec{F}\cdot d\vec{r}_2$$

for all C_1, C_2 joining A, B

$$\Updownarrow$$

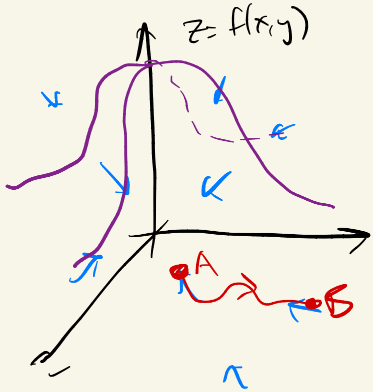
$\vec{F}$ has the "loop property"



$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

for all loops C

$$\Updownarrow$$



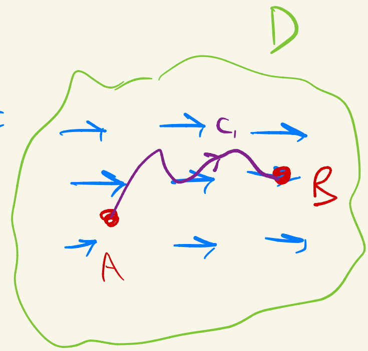
$\vec{F} = \nabla f$ for potential $f(x,y,z)$.

Fund thm line integrals (grad thm)

$$\int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$$

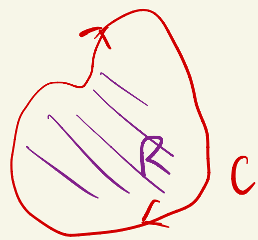
Fund Thm Line integrals:

$$\int_{C_1} \nabla f \cdot d\vec{r} = \underbrace{f(B) - f(A)}_{\text{boundary of } C_1}$$



Theme:

$$\iint_R \text{"derivative } F\text{"} \cdot dA = \int_C \text{"} F \cdot d\vec{r}\,\text{"}$$



$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Today: Green's theorem

$$\iint_R \big(\underbrace{\nabla \times \vec{F}}_{\text{curl } \vec{F}}\big) \cdot \hat{k} \, dx\,dy = \oint_C \vec{F} \cdot \vec{T} \, ds$$

$$\iint_R \big(\underbrace{\nabla \cdot \vec{F}}_{\text{div } \vec{F}}\big) \, dx\,dy = \oint_C \vec{F} \cdot \vec{n} \, ds$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}$$

Need some motivation for curl form of Green's thm.

Our goal: curl measures the "rotation" in a vector field

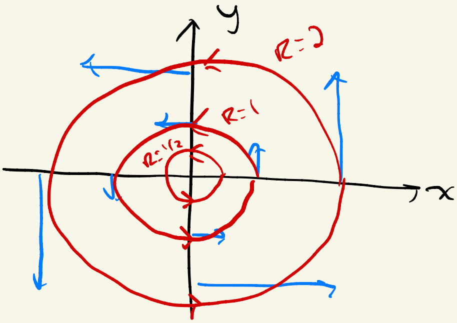
we can use rotation to "gain" circulation.

Component test says $\vec{F}$ conservative (so $\vec{F} = \nabla f$) iff $\text{curl}\,\vec{F} = 0$.

Larger circle: more fluid affected. and line integral thm.

Recall: Whirlpool

$\vec{F} = -y\hat{i} + x\hat{j}$



$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(0) - \cancel{\frac{\partial}{\partial z} x}^{\,0} \right)\hat{i}$$

$$- \left(\frac{\partial}{\partial x}(0) - \cancel{\frac{\partial}{\partial z}(-y)}^{\,0} \right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) \right)\hat{k}$$

$$= 2\hat{k}$$

curl $\vec{F}$ for any $\vec{F} = M\hat{i} + N\hat{j} + 0\hat{k}$ will point in $\hat{k}$ direction

$+\,\text{curl}\,\vec{F}\cdot\hat{k}$ means CCW rotation

$-\,\text{curl}\,\vec{F}\cdot\hat{k}$ means CW rotation

$C: \quad \vec{r}(t) = R(\cos t\,\hat{i} + \sin t\,\hat{j})$

$\text{Area} = \pi R^2$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (-R\sin t\,\hat{i} + R\cos t\,\hat{j}) \cdot (-R\sin t\,\hat{i} + R\cos t\,\hat{j})\,dt$$

$$= \int_0^{2\pi} R^2(\sin^2 t + \cos^2 t)\,dt$$

$$= 2\pi R^2$$

$$\shortparallel$$

$$\text{Area}\cdot(\text{curl}\,\vec{F}\cdot\hat{k})$$

circulation grows like Area grows when $\text{curl}\,\vec{F}\cdot\hat{k}$ is constant.

Another ex: Shearing flow

$\vec{F}(x,y) = y\hat{i}$

$$\text{curl}\,\vec{F} = \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(y)\right)\hat{k}$$

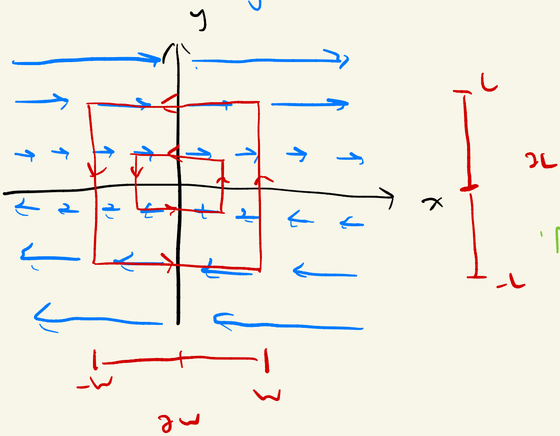
$$= -\hat{k}$$

CW rotation

Note: curl is constant

this gives curl since top of field is stronger

(like canoe where person on right paddles stronger — we spin!)



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy$$

$-dx$

dx

$\vec{F} \perp d\vec{r}$

$$= \int_{top} y(-dx) + \int_{bottom} y\,dx + \cancel{\int_{left} \vec{F}\cdot d\vec{r}} + \cancel{\int_{right} \vec{F}\cdot d\vec{r}}$$

$$= \int_{-W}^{W} -L\,dx + \int_{-W}^{W} -L\,dx$$

$$= -L(2W) - L(2W)$$

$$= -4LW$$

$$= (\text{curl}\,\vec{F}\cdot\hat{k})\cdot \text{Area}$$

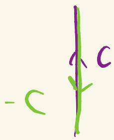
We can prove this always works if curl $\vec{F}$ is constant or roughly constant.

How to generalize?

Well... we can break line integrals into useful pieces

<u>Recall:</u>

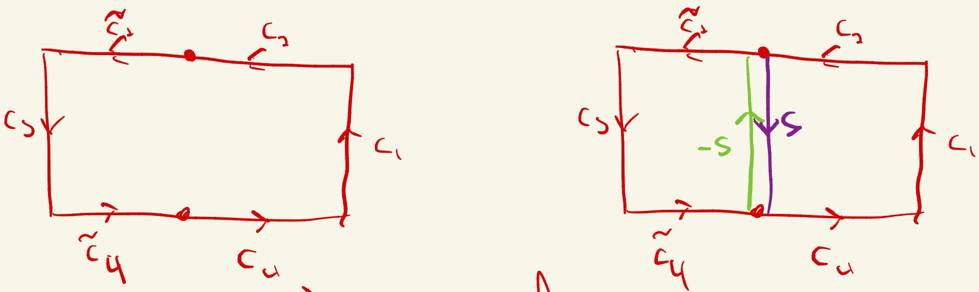
$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = 0$$



$$\int_C \vec{F}\cdot d\vec{r} + \int_{-C} \vec{F}\cdot d\vec{r} = \int_{C-C} \vec{F}\cdot d\vec{r}$$

$$= 0$$

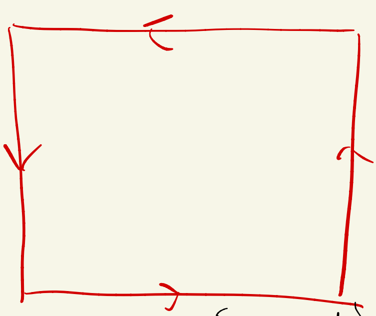
$$\oint_C \vec{F}\cdot d\vec{r}$$

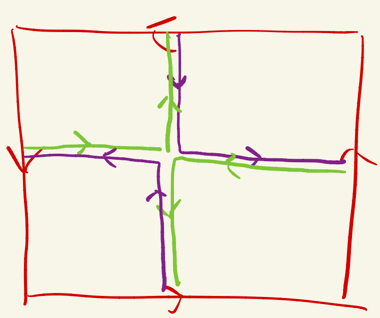


$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{c_1+c_2+\tilde{c}_2+c_3+\tilde{c}_4+c_4} \vec{F}\cdot d\vec{r} = \oint_{(c_1+c_2+s+c_4)+(\tilde{c}_2+c_3+\tilde{c}_4-s)} \vec{F}\cdot d\vec{r}$$

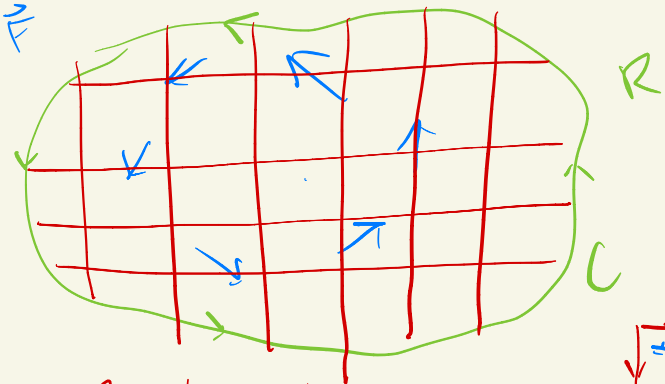
adding 0 = s − s

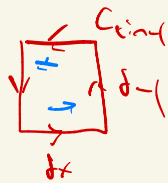
So any region we want can be broken down into tiny ones...





Same line integrals!





Break into boxes,

... in each box, if area tiny enough curl $\vec F \approx$ constant

here, in each box,

$$\oint_{C_{tiny}} \vec F\cdot d\vec r = \text{Curl}\,\vec F\cdot\hat k\,dx\,dy$$

Suggested (but not proven) by earlier calculations

Add them all up:

Green's theorem (Tangential form or circulation-curl form)

If C is a piecewise smooth oriented simple closed curve enclosing a region R, and $\vec F$ has continuous partials for each component, then

$$\oint_C \vec F\cdot d\vec r = \iint_R \text{curl}\,\vec F\cdot\hat k\, dA$$

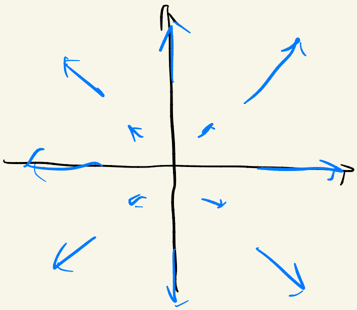
$$\oint_C \vec F\cdot d\vec r = \oint_C \vec F\cdot\vec T\,ds = \iint_R \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\,dx\,dy$$

What about flux?

Recall: Source and sink classics

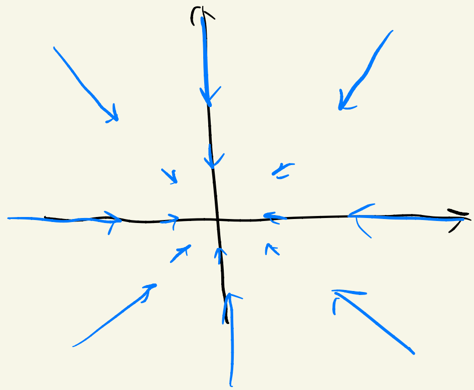
Source

$\vec{F} = x\hat{i} + y\hat{j}$



Sink

$\vec{F} = -x\hat{i} - y\hat{j}$



How to detect?

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (M, N)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

← Note - divergence is a scalar

$$\operatorname{div}\left(x\hat{i} + y\hat{j}\right) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y)$$

$$= 2$$

← Expanding (pushing out) $\operatorname{div}\vec{F} > 0$

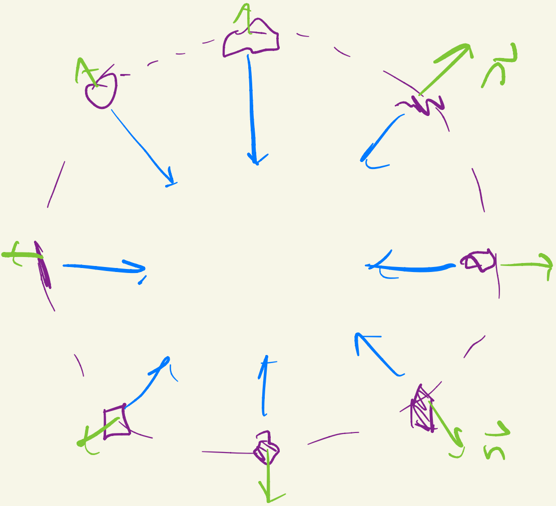
$$\operatorname{div}\left(-x\hat{i} - y\hat{j}\right) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y)$$

$$= -2$$

← Shrinking / sucking in $\operatorname{div}\vec{F} < 0$

Why should this be related to flux?

Recall Star wars attack of the clones



Interstellar objects

sucked in by

$\vec{F}$ - gravity

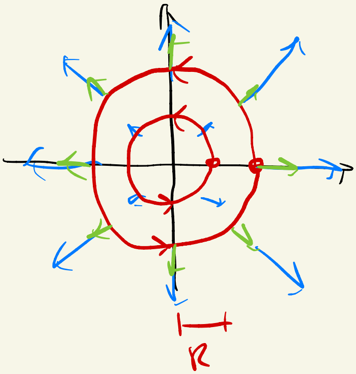
$$\oint_{\text{Interstellar objects}} \vec{F} \cdot \vec{n} < 0$$

getting sucked in!

$\operatorname{div} \vec{F} < 0$!

How is this line integral related to $\operatorname{div} \vec{F}$? How does it scale with size?

Ex: $\vec{F} = x\hat{i} + y\hat{j}$



Extra credit today:

$\vec{n}$ for circle $= \frac{1}{R}(x\hat{i} + y\hat{j})$

$\frac{1}{\sqrt{x^2+y^2}}(x\hat{i} + y\hat{j})$

since $\vec{n} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{R}$

$\vec{r}(t) = R(\cos t\,\hat{i} + \sin t\,\hat{j})$ $\qquad |\vec{r}\,'(t)|$

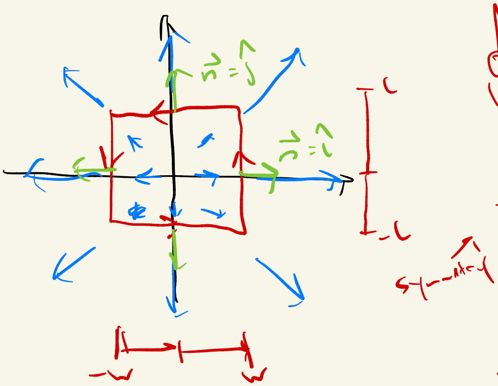
$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} (R\cos t\,\hat{i} + R\sin t\,\hat{j})\cdot(\cos t\,\hat{i} + \sin t\,\hat{j})\,R\,dt$$

$$= \int_0^{2\pi} R^2(1)\,dt$$

$$= 2\pi R^2$$

$$= (\text{div}\,\vec{F})\,\text{Area}$$

$\vec{F} = x\hat{i} + y\hat{j}$



-dx = forward dx direction backwards

$$\oint_C \vec{F}\cdot\vec{n}\,ds$$

$$= 2\cdot\int_{top} \vec{F}\cdot\vec{n}\,ds + 2\cdot\int_{right} \vec{F}\cdot\vec{n}\,ds$$

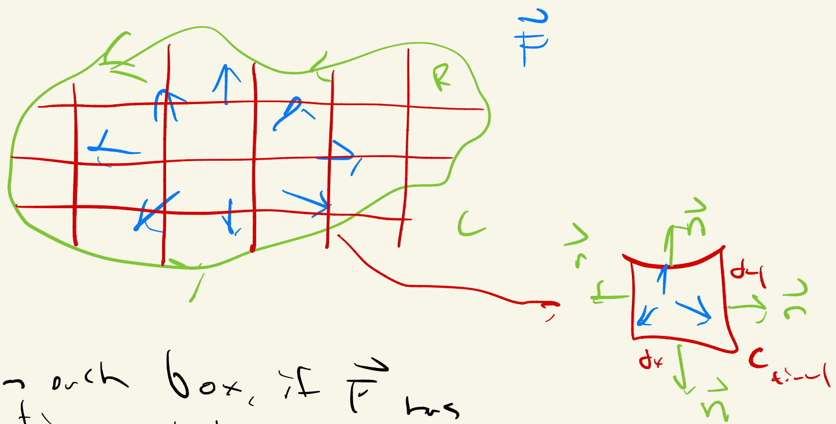
$$= 2\int_{-W}^{W} (x\hat{i} + L\hat{j})\cdot\hat{j}\,(-dx)$$

$$+ 2\int_{-L}^{L} (W\hat{i} + y\hat{j})\cdot\hat{i}\,dx$$

$(\text{div}\,\vec{F})\,\text{Area}$ $= 2(2WL) + 2(2LW)$

$= 2\cdot 4LW$

How to generalize? Same procedure:



In each box, if $\vec{F}$ has continuous partials and box is small enough, $\operatorname{div}\vec{F} \approx$ constant so calculations suggest

$$\oint_{C_{final}} \vec{F}\cdot\vec{n}\,ds = \operatorname{div}\vec{F}\,dA$$

Green's Theorem (Normal form or Flux-divergence form)

If C is a piecewise smooth oriented simple closed curve enclosing a region R, and $\vec{F}$ has continuous partials for each component, then

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA$$

$$= \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dx\,dy$$

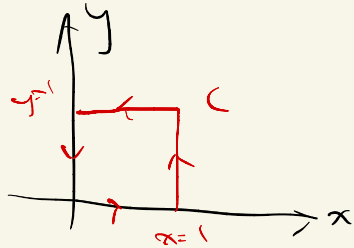
Powerful:

Now can change between line integrals and area integrals.

Ex:

Evaluate the line integral

$$\oint_C xy\,dy - y^2\,dx$$



where C is square

Soln:

Could just evaluate, or: 2 options for Green's thm.

① Tangential form of some vector field

Let $\vec{F} = -y^2\hat{i} + xy\hat{j}$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C -y^2\,dx + xy\,dy$$

Green's
R component curl $\vec{F}$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \iint_R y - (-2y)\,dx\,dy$$

$$= \int_0^1\int_0^1 3y\,dx\,dy$$

$$= \frac{3}{2}$$

② Normal Form

$\vec{F} = xy\,\hat{i} + y^2\,\hat{j}$

div $\vec{F}$

$$\oint_C xy\,dy - y^2\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

"Flux form"

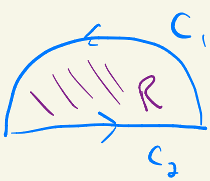
Green's Thm for flux-div

$$= \iint_R y + 2y\,dx\,dy$$

$$= \int_0^1 \int_0^1 3y\,dx\,dy$$

$$= \frac{3}{2}$$

Geometry Tricks (rel. to PS 8)

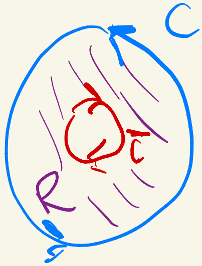


$$\oint_{C_1+C_2} \vec{F}\cdot d\vec{r} = \oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r}$$

Get flexibility in clever integrals

$$\oint_{C_1} \vec{F}\cdot d\vec{r} + \oint_{C_2} \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\cdot\hat{k}\,dx\,dy$$

$$\oint_{C_2} \vec{F}\cdot d\vec{r} = \iint_R \operatorname{curl}\vec{F}\cdot\hat{k}\,dx\,dy - \oint_{C_1} \vec{F}\cdot d\vec{r}$$



$$\iint_R \operatorname{curl} \vec{F}\cdot\hat{k}\,dx\,dy = \oint_C \vec{F}\cdot d\vec{r} + \oint_{\bar{C}} \vec{F}\cdot d\vec{r}$$

oriented CW = −CCW

Next up: surface parametrizations and surface integrals.

Midterm:

Ch. 13 (whatever done)

Parametrization is crucial!

Ch. 16.1–16.3 (no Green's thm) (not even allowed to use it)

40 min/part, 10 min upload time/part